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Some results on the asymptotic behaviour of coefficients of large powers of functions

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Abstract

We review existing results on the asymptotic approximation of the coefficient of order n of a function $f(z)^d$, when n and d grow large while staying roughly proportional Then we present extensions of these results to allow more general relationships between n and d and to take into account a multiplicative factor $\psi(z)$, that may itself include 'large' powers of simpler functions. A common feature of all the results of the paper is the use of a saddle point approximation; in particular we show that an approximate saddle point can give simpler results, and we characterize precisely how far from the exact value this approximate saddle point can be.

1. Introduction

Generating functions of the type $\phi(z)=f(z)^d$, where f is a given function with positive coefficients and d is a parameter that tends to infinity, appear in several problems of discrete probability theory, combinatorial enumeration, etc. These problems often require an estimate of the nth coefficient of f^d, which we denote by $[z^n]\{f(z)^d\}$, for large n and d.

For example, let $X_1, \ldots, X_d$ be d random variables, independent and with the same probability distribution defined by the generating function $f(z)$. Their sum $S_d=\sum_{i=1}^d X_i$ has for generating function $f^d(z)$, whose coefficient of order n is the probability $\Pr(S_d=n)$. The average value of S_d is $df'(1)$, and its variance is also of order d. The situations where $n=df'(1)+o(\sqrt{d})$, $n=o(d)$ or $d=o(n)$ describe the behaviour of the sum respectively close to the mean (in a range where the central limit theorem applies), before or beyond the mean (in an area of large deviations).

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Coefficients of the type $[z^n]\{f^d(z)\}$ appear in asymptotic coding theory [11], in the evaluation of some parameters on forests of trees [18, 23], in the evaluation of diagonal coefficients of some bivariate functions $F(z,u)$, and in a class of asymptotic distributions related to urn models, that require computing the coefficient $[y^n]\{f(x,y)^d\}$ of a bivariate function: see [15, 16] for a survey of results on urn models and [10, 8] for some applications to relational database theory. Related problems also appear in the evaluation of some trie parameters [6] or of the number of lattice points in a ball [17, 22], and in the analysis of a random walk on a hypercube [4].

This paper is intended as a survey of results on the asymptotic estimation of coefficients of the type $[z^n]\{f^d(z)\}$; it also presents and proves some as yet unpublished results. A preliminary version was presented in [9]. One of the major goals of this paper is to present the relevant results, old and new, in a unified and easy-to-use manner; thus Section 2 begins by introducing some notations and recalls the basis of the main technique (a saddle point approximation). Former results pertaining to the asymptotic approximation of $[z^n]\{f(z)^d\}$, for large n and d growing at a similar rate, are presented in Section 3. We then extend these results to allow for different growth rates for n and d. In Section 4, we allow for a multiplicative factor $\psi(z)$ and study the coefficient $[z^n]\{f(z)^d\psi(z)\}$. In particular, we prove that this function ψ can itself be a 'large' power ($\psi = g^{d_1}$), or product thereof, as long as its exponent d_1 does not grow too fast with respect to n and d. We indicate some applications, mostly related to urn models and Stirling numbers, in Section 5, and possible extensions of our work in Section 6. Finally we prove the new results of Sections 3 and 4 in Section 7.

2. Notations and methods

2.1. Notations

We consider in this paper functions of one variable that have a power series expansion: $f(z)=\sum_{k\geqslant 0} f_k z^k$. We assume in the sequel that the function f satisfies the following property.

Assumption $\mathscr{A}_1$. *The function f has real positive coefficients with $f_0 \neq 0$ and $f_1 \neq 0$, and a strictly positive, possibly infinite, radius of convergence R.*

As a consequence, the coefficients of the function f are such that $\mathrm{GCD}\{k: f_k \neq 0\}=1$, and there exists no entire function g and no integer $m \geqslant 2$ such that $f(z)=g(z^m)$; this fact will be used later on (in the proof of Lemma 3, Section 7.4). The condition on f_0 simply means that when $f(z)$ has valuation p, we can factor out z^p: If $f(z)=z^p(f_0+f_1 z+\cdots)$, then $f^d(z)=z^{dp}(f_0+f_1 z+\cdots)^d$. The restriction on f_1 is a technical one, which might be removed, but this implies more restrictive conditions on the relative growths of n and d than those given in some theorems of this paper.

To simplify the notations in the sequel, we define two operators on a function f:

$$\Delta f(z)=z\frac{\mathrm{d}}{\mathrm{d}z}\log f(z)=z\frac{f'}{f}(z); \qquad \delta f(z)=\frac{f''}{f}(z)-\frac{f'^2}{f^2}(z)+\frac{f'(z)}{zf(z)}.$$

These operators are related by: $z\delta f(z)=(\Delta f)'(z)$. When the coefficients of the function f are real and positive, we can check that, for all real positive z smaller than the radius of convergence of f, *the value of $\delta f(z)$ is strictly positive and the function Δf is increasing.* The terms $\Delta f(z)$ and $\delta f(z)$ have an interesting interpretation as the mean and variance of a probability distribution: Assume z is a fixed parameter, and define a random variable X by Proba$(X=k)=f_k z^k/f(z)$. Then the average value of X is $E[X]=\Delta f(z)$, and its variance is $\sigma^2(X)=z^2\delta f(z)$.

2.2. *Coefficients of an analytic function*

Before studying a function $f^d(z)$, we first recall results valid for any analytic function ϕ with positive coefficients. Its coefficient of order n is given by Cauchy's formula, where the integration contour is a closed curve around the origin of the complex plane, inside the convergence domain:

$$[z^n]\phi(z)=\frac{1}{2i\pi}\oint\phi(z)\frac{\mathrm{d}z}{z^{n+1}}.$$

We immediately deduce from it an upper bound $|[z^n]\phi(z)|\leqslant(1/2\pi)\oint|\phi(z)z^{-n-1}|\,\mathrm{d}z$. Integrating on a circle of radius ρ smaller than the radius of convergence of ϕ gives

$$|[z^n]\phi(z)|\leqslant\phi(\rho)\rho^{-n}, \tag{1}$$

and the best (smallest) upper bound is obtained, when possible, for ρ such that $\rho\phi'(\rho)/\phi(\rho)=n$.

For example, let us assume that ϕ is the generating function of a random variable X, of mean μ, and let $n=(1+\delta)\mu$. Then $\Pr(X=n)=[z^n]\phi(z)$ is bounded from above by $\phi(\rho)\rho^{-n}$. Setting $\rho=\mathrm{e}^t$ and using the fact that $\phi(\mathrm{e}^t)=E(\mathrm{e}^{tX})$, we get

$$\Pr(X=(1+\delta)\mu)\leqslant\frac{E(\mathrm{e}^{tX})}{\mathrm{e}^{t(1+\delta)\mu}}.$$

This is *Chernoff's bound*, which often gives useful information on the probability that a random variable is at distance at least $1+\delta$ of its mean.

Now assume that the random variable X is itself obtained by summing d independent random variables with a common distribution: $\phi(z)=f^d(z)$. Then we have that $\Pr(X=n)\leqslant f^d(\rho)\rho^{-n}$, and this bound is tightest for ρ such that $\rho f'(\rho)/f(\rho)=n/d$.[1]

[1] The tail probability usually studied in probability theory is $\Pr(X\geqslant n)$ rather than $\Pr(X=n)$. However, these two probabilities are equivalent, as can be seen, for example, in Feller's [5, p. 192] treatise.

2.3. The saddle point approximation

The upper bound (1) can be refined to give an approximation of $[z^n]\,\phi(z)$: Instead of bounding ϕ on the integration circle, we look closely at the points that give the main contribution to the integral. This is the basis of the saddle point method (see for example [3] for a general presentation and [14, 24 Ch. 5, 25] for applications to the approximation of generating function coefficients). It turns out that, if we can choose for radius of the integration circle the point ρ defined by the equation $\rho\phi'(\rho)/\phi(\rho)=n$, the main part of the integral often comes from the vicinity of ρ, which is a *saddle point*. Define $h(z)=\log\phi(z)-(n+1)\log z$; we get

$$[z^n]\,\phi(z)=\frac{1}{2i\pi}\oint e^{h(z)}\,dz=\frac{e^{h(\rho)}}{\sqrt{2\pi h''(\rho)}}(1+o(1))=\frac{\phi(\rho)}{\rho^{n+1}\sqrt{2\pi h''(\rho)}}(1+o(1)).$$

This approximation is known to hold for a large class of functions $\phi(z)=f^d(z)$, when n/d belongs to an interval $[a,b]$ $(0<a<b)$ and $n,d\to+\infty$ [2, 12, 13].

The application of the saddle point method to the asymptotic evaluation of coefficients of a function is closely related to Laplace's method for approximating an integral. Although this method is usually applied to integrals depending on one parameter, Fulks [7] and Pederson [21] have studied integrals depending on two parameters, that are in the same vein as the problem of evaluating $[z^n]\{f^d(z)\}$, where we have two parameters n and d.

3. Asymptotic approximations of coefficients

3.1. The case n constant

We include this case for the sake of completeness, although it presents no real difficulty. If n is constant, the saddle point method does not work; however a direct analysis can give some information. For example, the following result simply means that the first coefficients of $f(z)^d$ behave as those of $(f_0+f_1 z)^d$.

Theorem 1. *If the function f has real positive coefficients, such that $f_0\neq 0$ and $f_1\neq 0$, then, for $d\to+\infty$ and for any fixed n,*

$$[z^n]\{f^d(z)\}=\binom{d}{n}f_0^{d-n}f_1^n(1+O(1/d)).$$

Theorem 1 is proved by writing the desired coefficient as a sum of (a fixed number of) multinomial coefficients, which are themselves easily approximated. If we allow n to grow, both the number of terms in the sum and the terms themselves are unbounded, and this proof no longer holds.

3.2. A general formula when $n=\Theta(d)$

The problem of finding the asyumptotic value of $[z^n]\{f(z)^d\}$, when $n,d\to+\infty$ and when n and d are roughly proportional, was studied for example by Daniels [2] and by Greene and Knuth [13], mostly for probability generating functions. As noted by Good [12], this result is actually valid for a larger class of functions, such as entire functions or functions defined on an open disk; moreover it can be improved to give further terms of an asymptotic development. We give below the main result [2, 12 p. 868].

Theorem 2. *Let f be a function satisfying Assumption $\mathcal{A}_1$ of Section 2.1, and let R be its radius of convergence. Assume that n/d belongs to an interval $[a,b]$, $0<a<b$, and that $n,d\to+\infty$. Define ρ and σ^2 by $\Delta f(\rho)=n/d$ and $\sigma^2=\rho^2\delta f(\rho)$. If $\rho<R$, then*

$$[z^n]\{f(z)^d\}=\frac{f(\rho)^d}{\sigma\rho^n\sqrt{2\pi d}}(1+o(1)).$$

We can simplify Theorem 2 further if n/d has a finite, non-null, limit.

Corollary 1. *Under the assumptions of Theorem 1, if there exist two real strictly positive constants k and m such that $n=kd+m$, then*

$$[z^n]\{f(z)^d\}=\frac{A^d}{B\rho_0^m\sqrt{d}}(1+o(1)),$$

for suitable constants $A=f(\rho_0)\rho_0^{-k}$ and $B=\sigma\sqrt{2\pi}$, and with ρ_0 the solution (independent of n and d) of $\Delta f(z)=k$. Note that σ too is a constant: $\sigma^2=\rho_0^2\delta f(\rho_0)$. If $n=kd$, i.e. $m=0$, then we have the simpler formula

$$[z^n]\{f(z)^d\}=\frac{A^d}{B\sqrt{d}}(1+o(1)).$$

It is possible to get some information on the variation of the term A when the quotient n/d is finite and bounded away from 0. *Let $A=A(k)$ with $k=n/d$. On a closed interval of $]0,+\infty[$ including $\Delta f(1)$, $A(k)$ is a unimodal function of k, first increasing then decreasing, with a maximum $A(\Delta f(1))=f(1)$.*

3.3. Function defined by an implicit equation

A recent paper by Meir and Moon [18] deals with the approximation of the coefficient of z^n in $f(z)^d$, when $d,n\to+\infty$ and $d=O(n)$, and with f defined by an implicit equation: $f(z)=z\phi(f(z))$ and $f(0)=0$. This improves on a former result by Flajolet and Steyaert [23], which was proved for $d=o(n)$, more precisely for $d\leqslant\sqrt{n}/\log^3(n)$. Meir and Moon give the following result.

Theorem 3. *Let ϕ be a function satisfying Assumption $\mathscr{A}_1$ of Section 2.1 and define a function f by $f(z)=z\phi(f(z))$ and $f(0)=0$. Let $d=\alpha n+\lambda\sqrt{n}+o(\sqrt{n})$, with α a constant such that $0\leqslant\alpha<1$ and $(\Delta\phi)^{-1}(1-\alpha)$ exists, and with λ a finite (positive or negative, possibly null) constant. Then, for n, $d\to+\infty$*

$$[z^n]\{f(z)^d\}=\frac{d}{n\sigma\sqrt{2\pi n}}e^{-\lambda^2/2\sigma^2}\rho^{d-n}\phi(\rho)^n(1+o(1)),$$

where ρ is defined by $\Delta\phi(\rho)=1-\alpha$ and σ^2 by

$$\sigma^2=\rho^2\frac{\phi''(\rho)}{\phi(\rho)}+\alpha(1-\alpha)=\rho^2\delta\phi(\rho).$$

Meir and Moon actually prove in passing the following result:

$$[t^n]\{\phi(t)^d\}=\frac{e^{-\lambda^2/2\sigma^2}}{\sigma\sqrt{2\pi d}}\frac{\phi(\rho)^d}{\rho^n}(1+o(1)),$$

and their range of validity is for $n=(1-\alpha)d+\lambda\sqrt{d}+o(\sqrt{d})$. For $\alpha>0$, this is basically an extension of Theorem 2 (to allow $\lambda\neq 0$) applied to $n=kd+O(\sqrt{d})$ with a constant $k=1-\alpha$ in $]0,1[$. Theorem 3 is then obtained by an application of the Lagrange inversion formula:

$$[z^n]\{f(z)^d\}=\frac{d}{n}[t^{n-d}]\{\phi(t)^n\}.$$

When $\alpha=0$ but $\lambda>0$, Theorem 3 gives an approximation valid for $d=\lambda\sqrt{n}(1+o(1))$, i.e. $d^2=\lambda^2 n(1+o(1))$. If $\alpha=\lambda=0$, the result holds for $d=o(\sqrt{n})$, i.e. $d^2=o(n)$. This means that Meir and Moon have results for $d\approx\alpha n$ (when $\alpha\neq 0$) or for $d^2=O(n)$ (when $\alpha=0$). If $\lambda=0$ and $\alpha\neq 0$, and if we have $f(z)=z^q g(z)$ with $g(0)\neq 0$, then either one of Theorem 2 or Theorem 3 can be applied indifferently to evaluate the coefficient $[z^n]\{f^d(z)\}=[z^{n-qd}]\{g^d(z)\}$, for $d=\alpha n+o(\sqrt{n})$.

3.4. The case $n=o(d)$

We study now the case where d and n both grow large, but n stays much smaller than d. Theorem 4 is an extension of Theorem 2 to the case $n=o(d)$.

Theorem 4. *Let the function f satisfy Assumption $\mathscr{A}_1$ of Section 2.1 and let $n=o(d)$, with $n,d\to+\infty$. Define ρ as the unique real positive solution of $\Delta f(z)=n/d$. Then*

$$[z^n]\{f(z)^d\}=\frac{f(\rho)^d}{\rho^n\sqrt{2\pi n}}(1+o(1)).$$

Theorem 4 is proved by integrating on a circle going through the saddle point ρ, which becomes o(1) for $n=o(d)$. The singularities are beyond the integration contour as soon as n and d are large enough. The complete proof can be found in Section 7.6. below.

Theorem 4 is closely related to a result of Odlyzko and Richmond [20] that holds for the coefficient of z^n in a class of powers of polynomials f^d, when f is the generating function of a probability distribution with finite support, and for n and d such that, with q denoting the degree of f, $qd-n\to+\infty$. Their paper thus covers the case $n=o(d)$ and part of the case $n\Theta(d)$.

If we have more information on the respective orders of growth of n and d, we can obtain a useful approximation of the saddle point ρ and give a more precise form of Theorem 4. The following corollary, for example, can be used when $n=o(\sqrt{d})$ or $n=o(d^{2/3})$.

Corollary 2. *If f satisfies Assumption $\mathscr{A}_1$ of Section 2.1 and if $n=o(\sqrt{d})$, with $n, d\to+\infty$, then*

$$[z^n]\{f^d(z)\}=\frac{f_0^d}{\sqrt{2\pi n}}\left(\frac{\mathrm{e}f_1 d}{f_0 n}\right)^n(1+o(1))=\frac{f_0^d}{n!}\left(\frac{f_1 d}{f_0}\right)^n(1+o(1)).$$

If we only have the weaker condition $n=o(d^{2/3})$, then

$$[z^n]\{f^d(z)\}=\frac{f_0^d}{\sqrt{2\pi n}}\left(\frac{\mathrm{e}f_1 d}{f_0 n}\right)^n\exp\left(\frac{n^2}{d}\left(\frac{f_0 f_2}{f_1^2}-\frac{1}{2}\right)\right)(1+o(1)).$$

Intuitively, Corollary 2 means that, when $n=o(\sqrt{d})$, the first two coefficients of f determine the main term in the asymptotic expression of the coefficients of f^d. This result can be compared to the relevant one for an affine function (although an affine function does not satisfy Assumption $\mathscr{A}_1$): $[z^n]\{(f_0+f_1 z)^d\}=\binom{d}{n}f_0^{d-n}f_1^n$; Stirling's formula for the factorials gives an approximation equivalent to the first one of Corollary 2. When n increases with respect to d, the other coefficients are progressively introduced. As long as some relationship $n^l=o(d^q)$ holds with $l, q>1$, it is possible to get a result similar to Corollary 2. This requires a good approximation of the saddle point ρ, and might become quite involved according to which coefficients of f are null, but it would be possible to work it out for a given function f. However, if $n=d/(\log d)$ for example, we cannot find a relationship $n^l=o(d^q)$ with $l, q>1$ and we have to take all the coefficients of f into account.

3.5. The case $d=o(n)$

If the function f satisfies some functional equation, the result of Meir and Moon presented in Section 3.3 can sometimes be applied. More generally, we can prove analogs of Theorems 2 and 4 for some classes of functions, using similar technics (see Section 7 for proofs).

Theorem 5. *Let f be a function satisfying Assumption $\mathscr{A}_1$ of Section 2.1 and such that $f(z)=\mathrm{e}^{P(z)}$, where $P(z)=\sum_{0\leqslant j\leqslant q}p_j z^j$ is a polynomial of degree $q>1$ with positive coefficients. Let $n, d\to+\infty$ in such a way that $d=\mathrm{o}(n)$ but $(\log n)^{3q}n^{2q-3}=\mathrm{o}(d^{3(q-1)})$, and define ρ as the unique real positive solution of $zP'(z)=n/d$. Then*

$$[z^n]\{\mathrm{e}^{dP(z)}\}=\frac{\mathrm{e}^{dP(\rho)}}{\rho^n\sqrt{2\pi n}}(1+\mathrm{o}(1)).$$

Theorem 5 holds when $d=\mathrm{o}(n)$ and $n=\mathrm{o}(d\sqrt{d})$, whatever the degree of P. However, we cannot easily adapt it to deal with the exponential of an entire function $g(z)$: An important part of the proof relies on knowing the order of growth of the saddle point, which is defined by the equation $\Delta f(z)=n/d$; this is possible when $\log f$ is a polynomial, but cannot be extended when we only know that $\log f$ is an entire function.

If the function f is not entire, we have to take into account its singularities. For example, we can prove the following theorem for a meromorphic function with one single pole on its circle of convergence; this result holds as long as there exists some $k>1$ such that $n=\mathrm{O}(d^k)$.

Theorem 6. *Let f be a meromorphic function with positive coefficients, whose singularity of smallest modulus is a pole at 1 of order p: $f(z)=g(z)/(1-z)^p$, where g is a function analytic for $|z|\leqslant 1$ and with positive coefficients. Assume that $f_1\neq 0$, and define ρ by $\Delta f(\rho)=n/d$. Then, if $d=\mathrm{o}(n)$ and $\log(n/\sqrt{d})=\mathrm{o}(d^{1/3})$, we have*

$$[z^n]\{f^d(z)\}=\sqrt{\frac{pd}{2\pi}}\cdot\frac{f^d(\rho)}{n\rho^n}(1+\mathrm{o}(1)).$$

4. Introducing a factor $\psi(z)$

We now allow for a multiplicative factor $\psi(z)$ and study $[z^n]\{f^d(z)\psi(z)\}$. The function ψ may itself be defined in terms of d or of other parameters, as long as the following property is satisfied.

Assumption $\mathscr{A}_2$. *The function ψ has positive coefficients, such that $\psi(0)\neq 0$, has a strictly positive radius of convergence, and either is fixed, or is a finite product of 'not too large' powers of functions. In this case, it has the following form, where p is any fixed integer and the $d_j\to+\infty$:*

$$\psi(z)=\prod_{j=1}^{p}g_j(z)^{d_j}\quad \text{with } d_j=\mathrm{o}\left(\frac{d}{\sqrt{n}}\right),\quad 1\leqslant j\leqslant p. \tag{2}$$

In the theorems of this section, we shall use the abbreviated notation '$d_j \to +\infty$' to mean *if the function* ψ *satisfying assumption* $\mathscr{A}_2$ *is of the type* (2), *then* $d_j \to +\infty$ *for* $j=1,\ldots,p$, *in such a way that* $d_j = o(d/\sqrt{n})$.

Anticipating on the proofs of the theorems presented below, we can justify the condition (2) on ψ as follows. An extra factor $\psi(z)$ moves the saddle point away from the value ρ_0 obtained for f^d; this does not matter as long as the new saddle point ρ stays close enough, within $o(1/\sqrt{n})$ of ρ_0. The difference $\rho - \rho_0$ is $\Theta(\rho_0(\sum_i d_j/d))$, hence the condition (2).

We now present theorems which extend the former ones to allow an extra factor ψ. In the case where the function ψ has no extra parameters d_j, these theorems can be derived from the corresponding theorems of Section 3 by a result of Odlyzko, who gives upper and lower bounds for weighted sums of coefficients [19]. Theorem 7 is an obvious extension of Theorem 1, whose proof is left to the reader.

Theorem 7. *If f is a function with positive coefficients such that $f_0 \neq 0$ and $f_1 \neq 0$, and if the function ψ satisfies Assumption $\mathscr{A}_2$, then for n constant and $d, d_j \to +\infty$*

$$[z^n]\{f^d(z)\psi(z)\} = \binom{d}{n} f_0^{d-n} f_1^n \psi(0)(1 + O(1 + \sqrt{d})).$$

When n and d have the same growth rate, we can prove the following result, which was given in [11] for a function ψ satisfying the first part of Assumption $\mathscr{A}_2$, but not of the type (2). The proof of the present version is given in Section 7.11.

Theorem 8. *Let f satisfy Assumption $\mathscr{A}_1$ of Section* 2.1, *and let ψ be a function satisfying Assumption $\mathscr{A}_2$ of Section* 4. *Assume that the equation $\Delta f(z) = n/d$ has a real positive solution ρ smaller than the radii of convergence of f and of ψ. Then, for $n, d, d_j \to +\infty$ and $n = \Theta(d)$,*

$$[z^n]\{f^d(z)\psi(z)\} = \frac{f(\rho)^d \psi(\rho)}{\rho^{n+1}\sqrt{2\pi d\, \delta f(\rho)}}(1 + o(1)).$$

The case $n = o(d)$ is settled by the following theorem, whose proof can be found in Section 7.12.

Theorem 9. *Let f and ψ satisfy respectively Assumptions $\mathscr{A}_1$ and $\mathscr{A}_2$ and let $n = o(d)$, with $n, d, d_j \to +\infty$. Define ρ as the unique real positive solution of $\Delta f(\rho) = n/d$. Then*

$$[z^n]\{f(z)^d \psi(z)\} = \frac{f(\rho)^d \cdot \psi(\rho)}{\rho^n \sqrt{2\pi n}}(1 + o(1)).$$

If some relationship $n^l = o(d^q)$ holds, then we have the analog of Corollary 2.

Corollary 3. *If f satisfies Assumptions $\mathscr{A}_1$ of Section* 2.1, *if ψ satisfies $\mathscr{A}_2$ but with the stronger conditions $d_j = o(d/n)$, and if $n = o(\sqrt{d})$,*

$$[z^n]\{f^d(z)\psi(z)\} = \psi(0)\frac{f_0^d}{\sqrt{2\pi n}}\cdot\left(\frac{e f_1 d}{f_0 n}\right)^n (1+o(1)).$$

If we only have $n = o(d^{2/3})$, and $d_j = o(d/n)$, then

$$[z^n]\{f^d(z)\psi(z)\} = \psi(0)\frac{f_0^d}{\sqrt{2\pi n}}\left(\frac{e f_1 d}{f_0 n}\right)^n \exp\left(\frac{n^2}{d}\left(\frac{f_0 f_2}{f_1^2} - \frac{1}{2}\right)\right)(1+o(1)).$$

5. Some combinatorial applications

An easy check of our formulae is provided by the function $f(z) = e^z$. The saddle point is $\rho = n/d$; for d and n going to infinity in such a way that $n = o(d)$ and from Theorem 2, or for $d = \Theta(n) \to +\infty$ and from Theorem 4, we get

$$[z^n]\{e^{dz}\} = d^n/n! = \frac{e^n d^n}{n^n\sqrt{2\pi n}}(1+o(1)),$$

which is simply Stirling's formula for $n!$.

One of the basic constructions for obtaining combinatorial structures is to take a sequence of simpler objects. Let $f(z)$ be the generating function enumerating these objects according to their size; the generating function enumerating the sequences of d basic objects, according to their global size, is $f(z)^d$, and the coefficient $[z^n]\{f(z)^d\}$ eumerates the number of sequences of d basic objects of size n.

The same approach can also be used to analyse the abelian partitional complex, whose bivariate generating function has the form $\exp(xf(y))$. However, we usually do not count the structures of size 0 and we have $f(0) = 0$ and $n \geqslant d$. Let v be the valuation of f and define $f(y) = y^v g(y)$ with $g(0) \neq 0$; we have that $[z^n]\{f^d(z)\} = [z^{n-dv}]\{g^d(z)\}$. The results presented in Section 3 can now be applied to evaluate the number of composed objects of size $n \geqslant d$ which are a sequence of d simpler objects.

Classical examples are the Stirling numbers of the first and the second type. Stirling numbers of the first type enumerate, among other things, the number of permutations of n objects with k cycles. Their exponential generating function is $\sum_{n,k} s_{n,k} x^k y^n/n! = \exp(x \log(1/(1-y)))$; hence

$$s_{n,k} = \frac{n!}{k!}[y^{n-k}]\{f(y)^k\} \quad \text{with} \quad f(y) = \frac{1}{y}\log\frac{1}{1-y} = \sum_{n\geqslant 0}\frac{y^n}{n+1}.$$

For example, we can get an asymptotic equivalent for $n = k + o(k)$, or equivalently $k = n - o(n)$, but still $n - k \to +\infty$. The saddle point ρ is approximately $2(n-k)/k$

and Corollary 2 gives, for $n=k+o(\sqrt{k})$,

$$s_{n,k}=\frac{n!}{k!\sqrt{2\pi(n-k)}}\left(\frac{ek}{2(n-k)}\right)^{n-k}(1+o(1)),$$

i.e., using Stirling's approximation for $(n-k)!$ backwards,

$$s_{n,k}=\binom{n}{k}(k/2)^{n-k}(1+o(1)).$$

Let $S_{n,k}$ be a Stirling number of second type, enumerating for example the number of partitions of n objects into k blocks. These numbers have for exponential generating function $\sum_{n,k} S_{n,k}x^k y^n/n! = \exp(x(e^y-1))$, hence

$$S_{n,k}=\frac{n!}{k!}[y^{n-k}]\{f(y)^k\} \quad \text{with } f(y)=\frac{e^y-1}{y}=\sum_{n\geq 0}\frac{y^n}{(n+1)!}.$$

For $n=k+o(\sqrt{k})$ and $n-k\to+\infty$, Corollary 2 applied to $f(y)$ gives

$$S_{n,k}=\frac{n!}{k!\sqrt{2\pi(n-k)}}\left(\frac{ek}{2(n-k)}\right)^{n-k}(1+o(1))=\binom{n}{k}(k/2)^{n-k}(1+o(1)).$$

This asymptotic expression, which is also given for example in [1, p. 825], is the same as the one for the Stirling numbers of the first type. From Corollary 2, only f_0 and f_1 are important if $n-k=o(\sqrt{k})$. However, if the difference $n-k$ is of order at least $\sqrt{k}$, the next coefficients become important. For example, if $n-k\to+\infty$ with only $n-k=o(k^{2/3})$, then the second part of Corollary 2 shows that the Stirling numbers of the first and second type have a different behaviour:

$$s_{n,k}=\binom{n}{k}(k/2)^{n-k}e^{(n-k)^2/6k}(1+o(1));$$

$$S_{n,k}=\binom{n}{k}(k/2)^{n-k}e^{-(n-k)^2/6k}(1+o(1)).$$

Stirling numbers of the second type also appear in a classical occupancy problem of discrete probability theory: *We throw n balls into k urns randomly and independently; what is the number of urns with at least one ball?* Let $N_{n,d}$ be the number of ways of assigning the n balls to exactly d urns. If the balls are undistinguishable and if the urns have unbounded capacity, the associated generating function is [15]

$$\Phi(x,y)=\sum_{n,d}N_{n,d}x^d\frac{y^n}{n!}=(1+x(e^y-1))^k.$$

Let $f(y)=(e^y-1)/y$; we have $N_{n,d}=n!\binom{k}{d}[y^{n-d}]\{f(y)^d\}$, which can be expressed using Stirling numbers of the second type: $N_{n,d}=d!\binom{k}{d}S_{n,d}$.

6. Possible extensions

In this paper, we have recalled former results, or presented new results, on the asymptotic approximation of coefficients of the type $[z^n]\{f^d(z)\}$, with applications, and on the asymptotic approximation of the coefficient $[z^n]\{f^d(z)\psi(z)\}$; examples of applications using such coefficients can be found in [11]. Possible extensions include:

- Allowing the second coefficient of f to be null: $f_1=0$. This corresponds to a function $f(z)=1+f_2z^2+\cdots$. Preliminary studies indicate that such an extension considerably restricts the respective ranges of n and d.
- Allowing $d=o(n)$ for more general functions than those considered in Section 3.5. Here again we may have to introduce further growth restrictions on n and d, depending on the singularities of the function $f(z)$; we may also have to use a technique more adapted to the nature of the singularities than the saddle point method.
- Removing the restriction that ψ has positive coefficients. This does not seem to pose any real difficulty, as opposed to the fact that the similar condition on f is essential.
- Obtaining further terms of an asymptotic expansion. This is similar to the extension of the results of Daniels [2] by Good [12], and should not introduce major difficulties.

7. Proofs of theorems

7.1. Intermediate results

We begin by stating a succession of intermediate results, from which we shall derive our theorems. We assume in all the lemmas and propositions below that $n, d \to +\infty$, although their respective rates may vary.

Lemma 1. *Let f be a function satisfying Assumption $\mathscr{A}_1$ of Section 2.1. Then the equation $\Delta f(z)=(n+1)/d$ has at most one real positive solution. Furthermore, when $n=o(d)$ this solution is asymptotically equal to $f_0(n+1)/(f_1 d)(1+O(n/d))$.*

Proof of Lemma 1. It suffices to check that $\Delta f(0)=0$ and that the function $\Delta f(z)$ is continuous and increasing for $z\in[0,+\infty[$; this holds as soon as the function f has positive coefficients. When we assume that $n=o(d)$, the solution of the equation $\Delta f(z)=(n+1)/d$ always exists and becomes smaller than any fixed number; solving the approximate equation obtained by substituting its Taylor expansion near the origin for $\Delta f(z)$ gives an approximation of the solution, which could be improved into an asymptotic expansion if desired. □

Lemma 2. *Let ρ be a strictly positive number, and assume that $h(z)$ is a function defined and analytic in an open set including the closed disk $\{|z|\leqslant\rho\}$, such that $h'(\rho)=0$ and*

$h''(\rho)$ is strictly positive. Assume also that there exists $\alpha_0 > 0$ such that $h(\rho e^{i\theta})$ has a Taylor expansion in terms of θ for $|\theta| \leqslant \alpha_0$:

$$h(\rho e^{i\theta}) = h(\rho) + \frac{\rho^2}{2} h''(\rho)(e^{i\theta}-1)^2 + O(\rho^3(e^{i\theta}-1)^3 \| h''' \|).$$

Let $\alpha_1 = \inf(\alpha_0, 0.9)$; then, for all α in $]0, \alpha_1[$

$$\frac{1}{2i\pi}\int_{|\theta|\leqslant\alpha} e^{h(\rho e^{i\theta})} d(\rho e^{i\theta}) = \frac{e^{h(\rho)}}{\sqrt{2\pi h''(\rho)}}(1 + O(\sqrt{h''(\rho)}\alpha^2 \rho e^{-\rho^2\alpha^2 h''(\rho)/4}) + O(\| h'''' \| \rho^3 \alpha^3) + O(e^{-\rho^2\alpha^2 h''(\rho)/4})).$$

We defer the proof of Lemma 2 until Section 7.3. (Lemma 2 is closely related to Lemma B of [8], which is given for a function with two parameters).

Lemma 3. *Let f be a function satisfying Assumption $\mathscr{A}_1$ of Section 2.1. Let ρ be a real positive number strictly smaller than the radius of convergence of f. Define $h(z) = d\log f(z) - (n+1)\log z$ and $C_\rho = f_0 f_1 \rho/(4f(\rho)(f_0 + f_1\rho))$. We have that*

$$\int_{\alpha\leqslant|\theta|\leqslant\pi} e^{h(\rho e^{i\theta})} d(\rho e^{i\theta}) = O(\rho e^{h(\rho)} e^{-C_\rho d\alpha^2}).$$

Lemma 3 is proved in Section 7.4.

Proposition 1. *Under the notations and Assumption $\mathscr{A}_1$ of Section 2.1, define the function h by $h(z) = d\log f(z) - (n+1)\log z$, and let $n, d \to +\infty$ in such a way that the equation $\Delta f(z) = (n+1)/d$ has a real positive solution ρ. Assume that f is analytic in a domain including the closed disk of radius ρ. Then there exists $\alpha_1 > 0$ such that, for any $\alpha \in]0, \alpha_1[$ and for $C_\rho = f_0 f_1 \rho/(4f(\rho)(f_0 + f_1\rho))$, we have that*

$$[z^n]\{f(z)^d\} = \frac{f(\rho)^d}{\rho^{n+1}\sqrt{2\pi d\cdot\delta f(\rho)}}(1 + O(\| h''' \| \rho^3\alpha^3) + O(\sqrt{h''(\rho)}\alpha^2 \rho e^{-\rho^2\alpha^2 h''(\rho)/4}) + O(e^{-\rho^2\alpha^2 h''(\rho)/4}) + O(\rho\sqrt{h''(\rho)} e^{-C_\rho d\alpha^2})).$$

Proposition 1 is proved in Section 7.2.

Proposition 2. *Under the assumptions of Proposition 1, let us choose for saddle point an approximate value $\rho = \rho_0(1+\eta)$, where ρ_0 is the exact saddle point defined by the equation $\Delta f(z) = (n+1)/d$. Define $\beta(z) = d^2/dz^2(\log f(z))$ and $\| \beta \| = \max\{\beta(z)\}$ for z between ρ_0 and ρ. Then we can substitute ρ for ρ_0 in Proposition 1, as long as η satisfies*

$$n\eta^2 = o(1) \quad \text{and} \quad d\| \beta \| \rho_0^2 \eta^2 = o(1).$$

The proof of Proposition 2 can be found in Section 7.5.

Proposition 3. *We require Assumptions $\mathscr{A}_1$ and $\mathscr{A}_2$ of Sections* 2.1 *and* 4. *Define the function h by* $h(z)=\mathrm{d}\log f(z)-(n+1)\log z+\log\psi(z)$. *For a suitable* $\alpha>0$, *for* ρ *defined by* $\Delta f(\rho)+\Delta\psi(\rho)/d=(n+1)/d$, *and for* $C_\rho=f_0 f_1\rho/(4f(\rho)(f_0+f_1\rho))$,

$$[z^n]\{f(z)^d\psi(z)\}=\frac{f(\rho)^d\psi(\rho)}{\rho^{n+1}\sqrt{2\pi h''(\rho)}}(1+\mathrm{O}(\|h'''\|\rho^3\alpha^3)$$
$$+\mathrm{O}(\alpha^2\rho\sqrt{h''(\rho)}\mathrm{e}^{-\rho^2\alpha^2h''(\rho)/4})$$
$$+\mathrm{O}(\mathrm{e}^{-\rho^2\alpha^2h''(\rho)/4})+\mathrm{O}(\rho\sqrt{h''(\rho)}\mathrm{e}^{-C_\rho d\alpha^2})).$$

Proposition 3 is proved in Section 7.4. As before, we can often use a simpler saddle point (see Section 7.10 for the proof of Proposition 4).

Proposition 4. *Under the assumptions of Proposition* 3, *let us choose for saddle point an approximate value* $\rho=\rho_0(1+\eta)$, *where* ρ_0 *is the exact saddle point defined by the equation* $\Delta f(z)+\Delta\psi(z)/d=(n+1)/d$. *Define* $\beta(z)=\mathrm{d}^2/\mathrm{d}z^2(\log f(z))$ *and* $\beta_j(z)=\mathrm{d}^2/\mathrm{d}z^2(\log g_j(z))$ *for* $1\leqslant j\leqslant p$. *Then we can substitute* ρ *for* ρ_0 *in Proposition* 3, *as long as* η *satisfies*

$$n\eta^2=\mathrm{o}(1);\quad d\beta(\rho_0)\rho_0^2\eta^2=\mathrm{o}(1);\quad d_j\beta_j(\rho_0)\rho_0^2\eta^2=\mathrm{o}(1)\quad(1\leqslant j\leqslant p).$$

7.2. Proof of Proposition 1

The proof of Proposition 1 is very similar to that of Theorem 2 given for example in [2]. However, Theorem 2 is valid for n and d proportional; we want to extend this when n and d both grow large, but are not required to be asymptotically proportional. Although the main term is the same, we need more precisions in the error terms, which we have to give in extenso.

Let $I_{n,d}=[z^n]\{f^d(z)\}$; from Cauchy's formula, we have that $I_{n,d}=(1/2\mathrm{i}\pi)\oint f^d(z)z^{-n-1}\,\mathrm{d}z$, where the integration contour is a simple curve around the origin. We shall integrate on a circle, whose radius is a saddle point. Define the function

$$h(z)=d\log f(z)-(n+1)\log z.$$

The equation giving the saddle point is $h'(z)=0$; we rewrite it as

$$\Delta f(z)=\frac{n+1}{d}. \tag{3}$$

From Lemma 1, this equation has a unique real positive solution ρ. The condition on the domain where f is analytic ensures that we can integrate f on the circle $z=\rho\mathrm{e}^{\mathrm{i}\theta}$. Hence we choose for integration contour a circle of radius ρ and center the origin:

$$I_{n,d}=\frac{1}{2\mathrm{i}\pi}\int_{\theta\in[-\pi,+\pi]}\mathrm{e}^{h(\rho\mathrm{e}^{\mathrm{i}\theta})}\,\mathrm{d}(\rho\mathrm{e}^{\mathrm{i}\theta}).$$

Let $\alpha \in]0, \pi[$, to be precised later on. We divide the integral in two parts: $I_{n,d} = I_1 + I_2$, with

$$I_1 = \frac{1}{2i\pi} \int_{|\theta| \leqslant \alpha} e^{h(\rho e^{i\theta})} d(\rho e^{i\theta}),$$

$$I_2 = \frac{1}{2i\pi} \int_{\alpha \leqslant |\theta| \leqslant \pi} e^{h(\rho e^{i\theta})} d(\rho e^{i\theta}).$$

We shall compute an approximation of I_1 using Lemma 2, and show by Lemma 3 that I_2 can be neglected. We can write a Taylor expansion of $h(\rho e^{i\theta})$ for θ close to 0: there exists some α_0 such that, for $|\theta| \leqslant \alpha_0$,

$$h(\rho e^{i\theta}) = h(\rho) + \rho h'(\rho)(e^{i\theta} - 1) + \frac{\rho^2}{2} h''(\rho)(e^{i\theta} - 1)^2 + O(\rho^3 (e^{i\theta} - 1)^3 \| h''' \|).$$

By definition of ρ, $h'(\rho) = 0$. Now the second derivative of h is

$$h''(z) = d\left(\frac{f''(z)}{f(z)} - \frac{f'^2(z)}{f^2(z)}\right) + \frac{n+1}{z^2}.$$

For ρ such that $\rho f'(\rho)/f(\rho) = (n+1)/d$, we have that

$$h''(\rho) = d\left(\frac{f''(\rho)}{f(\rho)} - \frac{f'^2(\rho)}{f^2(\rho)} + \frac{f'(\rho)}{\rho f(\rho)}\right) = d\delta f(\rho).$$

Hence $h''(\rho)$ is strictly positive and we can apply Lemma 2 to $h(z)$; we get

$$I_1 = \frac{e^{h(\rho)}}{\sqrt{2\pi h''(\rho)}} (1 + O(\| h''' \| \rho^3 \alpha^3) + O(\sqrt{h''(\rho)} \alpha^2 \rho e^{-\rho^2 \alpha^2 h''(\rho)/4}) + O(e^{-\rho^2 \alpha^2 h''(\rho)/4})), \tag{4}$$

where the main term can also be written as $f(\rho)^d / (\rho^{n+1} \sqrt{2\pi h''(\rho)})$.

We are now interested in the remaining term $I_2 = (1/2i\pi) \int_{\alpha \leqslant |\theta| \leqslant \pi} e^{h(z)} dz$, for $z = \rho e^{i\theta}$. From Lemma 3, we have that

$$\int_{\alpha \leqslant |\theta| \leqslant \pi} e^{h(\rho e^{i\theta})} d(\rho e^{i\theta}) = O(\rho e^{h(\rho)} e^{-C_\rho d \alpha^2}).$$

Hence

$$I_2 = \frac{e^{h(\rho)}}{\sqrt{2\pi h''(\rho)}} \cdot O(\rho \sqrt{h''(\rho)} e^{-C_\rho d \alpha^2}) \quad \text{with } C_\rho = \frac{f_0 f_1 \rho}{4 f(\rho)(f_0 + f_1 \rho)}. \tag{5}$$

Eqs. (4) and (5) can now be used to get a global evaluation of $I_{n,d}$, valid for any $\alpha \leqslant \alpha_1$:

$$I_{n,d} = \frac{e^{h(\rho)}}{\sqrt{2\pi h''(\rho)}} (1 + O(\| h''' \| \rho^3 \alpha^3) + O(\sqrt{h''(\rho)} \alpha^2 \rho e^{-\rho^2 \alpha^2 h''(\rho)/4}) + O(e^{-\rho^2 \alpha^2 h''(\rho)/4}) + O(\rho \sqrt{h''(\rho)} e^{-C_\rho d \alpha^2})).$$

7.3. Proof of Lemma 2

The function $h(\rho e^{i\theta})$, considered as a function of θ, has a Taylor expansion, around 0, for $|\theta| \leqslant \alpha_0$; we can assume without loss of generality that $\alpha_0 < \pi/2$. By definition of ρ, $h'(\rho)$ is null and we have for $z = \rho e^{i\theta}$ and $|\theta| \leqslant \alpha_0$,

$$h(z) = h(\rho) + \tfrac{1}{2}(z-\rho)^2 h''(\rho) + O(\| h''' \| (z-\rho)^3).$$

The error term depends on the value of h''' at some point between z and ρ.

The integral $I_1 = (1/2i\pi)\int_{|\theta| \leqslant \alpha} e^{h(\rho e^{i\theta})} d(\rho e^{i\theta})$ can be expressed as an integral on $\gamma = \{z = \rho e^{i\theta}, |\theta| \leqslant \alpha\}$:

$$I_1 = \frac{1}{2i\pi}\int_\gamma e^{h(z)} dz$$

$$= \frac{e^{h(\rho)}}{2i\pi}\int_\gamma e^{(1/2)(z-\rho)^2 h''(\rho) + O(\| h''' \| (z-\rho)^3)} dz$$

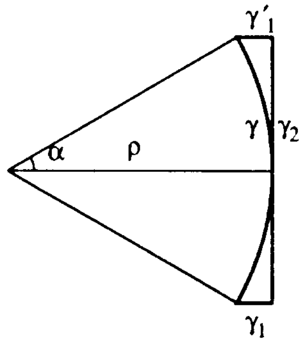
$$= \frac{e^{h(\rho)}}{2i\pi}(I'_1 + I''_1),$$

with

$$I'_1 = \frac{e^{h(\rho)}}{2i\pi}\int_\gamma e^{(1/2)(z-\rho)^2 h''(\rho)} dz,$$

$$I''_1 = \frac{e^{h(\rho)}}{2i\pi}\int_\gamma e^{(1/2)(z-\rho)^2 h''(\rho)}\left(e^{O(\| h''' \| (z-\rho)^3)} - 1\right) dz.$$

The integral $\int_\gamma e^{(1/2)(z-\rho)^2 h''(\rho)} dz$ can be approximated by a gaussian integral whose value is known. One way to do this consists in substituting the path $\gamma_1 \cup \gamma_2 \cup \gamma'_1$ to the arc γ: $\gamma_1 = \{z = \rho(1-u) - i\rho \sin\alpha\}$ and $\gamma'_1 = \{z = \rho(1-u) + i\rho \sin\alpha\}$ for $u \in [0, 1-\cos\alpha]$,



and $\gamma_2 = \{z = \rho + i\rho t\}$ for $|t| \leqslant \sin\alpha$. We then show that the integrals on γ_1 and γ'_1 give only error terms, and that the main part of I_1 comes from the integral on γ_2. On γ_1,

$z=\rho(1-u)-\mathrm{i}\rho\sin\alpha$ and $\mathrm{d}z=-\rho\mathrm{d}u$, which gives

$$\int_{\gamma_1} \mathrm{e}^{(z-\rho)^2 h''(\rho)/2}\,\mathrm{d}z=\rho\int_0^{1-\cos\alpha} \mathrm{e}^{\rho^2(u+\mathrm{i}\sin\alpha)^2 h''(\rho)/2}\,\mathrm{d}u. \tag{6}$$

The modulus of the right part of Eq. (6) can be bounded by

$$\rho \mathrm{e}^{-\rho^2 h''(\rho)(\sin^2\alpha)/2}\int_0^{1-\cos\alpha} \mathrm{e}^{\rho^2 h''(\rho)u^2/2}\,\mathrm{d}u.$$

Now

$$\int_0^{1-\cos\alpha} \mathrm{e}^{\rho^2 h''(\rho)u^2/2}\,\mathrm{d}u \leqslant (1-\cos\alpha)\mathrm{e}^{\rho^2 h''(\rho)(1-\cos\alpha)^2/2}=\mathrm{O}(\alpha^2 \mathrm{e}^{\rho^2 h''(\rho)(1-\cos\alpha)^2/2}).$$

This gives an upper bound on the integral on γ_1:

$$\int_{\gamma_1} \mathrm{e}^{(z-\rho)^2 h''(\rho)/2}\,\mathrm{d}z=\mathrm{O}(\rho\alpha^2 \mathrm{e}^{\rho^2((1-\cos\alpha)^2-\sin^2\alpha)h''(\rho)/2}).$$

Now $(1-\cos\alpha)^2-\sin^2\alpha=-\alpha^2+\mathrm{O}(\alpha^4)$, and we have an upper bound $(1-\cos\alpha)^2-\sin^2\alpha\leqslant-\alpha^2/2$ for $\alpha\leqslant 0.9$. Then the integral on γ_1 becomes $\mathrm{O}(\rho\alpha^2\mathrm{e}^{-\rho^2 h''(\rho)\alpha^2/4})$ as soon as $\alpha<\alpha_1$ with $\alpha_1=\inf(\alpha_0,\ 0.9)$. The integral on γ'_1 is computed in a similar way and gives the same bound.

We now compute the integral on γ_2:

$$\int_{\gamma_2} \mathrm{e}^{(z-\rho)^2 h''(\rho)/2}\,\mathrm{d}z=\mathrm{i}\rho\int_{|t|\leqslant\sin\alpha} \mathrm{e}^{-\rho^2 h''(\rho)t^2/2}\,\mathrm{d}t.$$

A change of variable $v=\rho\sqrt{h''(\rho)}\,t$ gives

$$\int_{\gamma_2} \mathrm{e}^{(z-\rho)^2 h''(\rho)/2}\,\mathrm{d}z=\frac{1}{\sqrt{h''(\rho)}}\int_{|v|\leqslant\rho\sqrt{h''(\rho)}\sin\alpha} \mathrm{e}^{-v^2/2}\,\mathrm{d}v,$$

which can be extended to a gaussian integral as soon as $\rho\sqrt{h''(\rho)}\sin\alpha\to+\infty$. We also use the fact that $\sin^2\alpha>\alpha^2/2$ for $\alpha\leqslant 1$, and thus for $|\alpha|\leqslant\alpha_1$, to simplify the error term $\mathrm{O}(\mathrm{e}^{-\rho^2 h''(\rho)\sin^2\alpha/2})$, which comes from extending the integral to the real axis, into $\mathrm{O}(\mathrm{e}^{-\rho^2 h''(\rho)\alpha^2/4})$:

$$\int_{|v|\leqslant\rho\sqrt{h''(\rho)}\sin\alpha} \mathrm{e}^{-v^2/2}\,\mathrm{d}v=\sqrt{2\pi}+\mathrm{O}(\mathrm{e}^{-\rho^2 h''(\rho)\alpha^2/4}).$$

Summing up, we get, for any $\alpha\leqslant\alpha_1$,

$$\int_{\gamma} \mathrm{e}^{(z-\rho)^2 h''(\rho)/2}\,\mathrm{d}z=\mathrm{i}\sqrt{\frac{2\pi}{h''(\rho)}}(1+\mathrm{O}(\mathrm{e}^{-\rho^2 h''(\rho)\alpha^2/4})$$

$$+\mathrm{O}(\rho\alpha^2\sqrt{h''(\rho)}\,\mathrm{e}^{-\rho^2 h''(\rho)\alpha^2/4})).$$

To complete the proof of Lemma 2, we now have to show that the term

$$I_1'' = \int_\gamma e^{(1/2)(z-\rho)^2 h''(\rho)} (e^{O(\|h'''\|(z-\rho)^3)} - 1)\, dz$$

gives a negligible contribution to the integral I_1. But

$$|I_1''| \leqslant \int_\gamma e^{(1/2)h''(\rho)\Re\{(z-\rho)^2\}} |e^{O(\|h'''\|(z-\rho)^3)} - 1| \cdot |dz|$$

$$= O(\|h'''\|\rho^3\alpha^3) \int_\gamma e^{(1/2)h''(\rho)\Re\{(z-\rho)^2\}} |dz|.$$

The computation of this last integral closely follows that of the integral $\int_\gamma e^{(1/2)h''(\rho)(z-\rho)^2} dz$, and we get that $I_1'' = I_1' O(\|h''\|\rho^3\alpha^3)$, which ends the proof of Lemma 2.

7.4. Proof of Lemma 3

We are interested here in getting an approximation for the integral $I_2 = (1/2i\pi)\int e^{h(z)} dz$, when $z = \rho e^{i\theta}$ and $\alpha \leqslant |\theta| \leqslant \pi$, and with $h(z) = d \log f(z) - (n+1)\log z$. From the definition of h, we have that

$$h(\rho e^{i\theta}) = h(\rho) + d(\log f(\rho e^{i\theta}) - \log f(\rho)) - (n+1)i\theta,$$

which shows that

$$I_2 = \frac{e^{h(\rho)}}{2\pi} \int_{\alpha \leqslant |\theta| \leqslant \pi} \rho e^{-in\theta} \left(\frac{f(\rho e^{i\theta})}{f(\rho)} \right)^d d\theta$$

and that

$$|I_2| \leqslant \frac{e^{h(\rho)}}{2\pi} \rho \int_{\alpha \leqslant |\theta| \leqslant \pi} \left| \frac{f(\rho e^{i\theta})}{f(\rho)} \right|^d d\theta. \tag{7}$$

Our problem is now to get an upper bound of $f(\rho e^{i\theta})$ valid on the set $\{\alpha \leqslant |\theta| \leqslant \pi\}$. From Assumption $\mathscr{A}_1$ of Section 2.1 (the function f has positive coefficients; hence the GCD of its coefficients is equal to 1), we deduce that $|f(\rho e^{i\theta})|$ attains its modulus $f(\rho)$ on the axis and nowhere else. More precisely, we can show that there exists a strictly positive factor C, independent of α, such that we have for all θ satisftying $\alpha \leqslant |\theta| \leqslant \pi$: $|f(\rho e^{i\theta})| \leqslant f(\rho)(1 - C\alpha^2)$. However, C depends on the radius ρ, so we shall denote it by C_ρ.

To get an expression of C_ρ, we use the assumption $f_1 \neq 0$, again coming from $\mathscr{A}_1$. Let $f(z) = \sum_{k \geqslant 0} f_k z^k$, where the f_k are real and positive; we have that

$$|f(\rho e^{i\theta})| = \left| \sum_k f_k \rho^k e^{ik\theta} \right| \leqslant |f_0 + f_1 \rho e^{i\theta}| + \left| \sum_{k \geqslant 2} f_k \rho^k e^{ik\theta} \right|.$$

The term $|\sum_{k\geqslant 2} f_k \rho^k e^{ik\theta}|$ is bounded by $\sum_{k\geqslant 2} f_k \rho^k$, which is equal to $f(\rho)-(f_0+f_1\rho)$. This shows that

$$|f(\rho e^{i\theta})| \leqslant |f_0+f_1\rho e^{i\theta}| + f(\rho)-(f_0+f_1\rho).$$

We next get an upper bound on $|f_0+f_1\rho e^{i\theta}|$:

$$|f_0+f_1\rho e^{i\theta}|^2 = (f_0+f_1\rho)^2(1-c_\rho(1-\cos\theta)),$$

with $c_\rho = 2f_0f_1\rho/(f_0+f_1\rho)^2$. The term c_ρ has an upper bound valid for all real ρ, but may become close to 0 when ρ itself is close to 0. Moreover, $1-c_\rho(1-\cos\theta)\leqslant(1-\cos\alpha)$ for $\alpha\leqslant|\theta|\leqslant\pi$. We deduce that

$$|f_0+f_1\rho e^{i\theta}| \leqslant (f_0+f_1\rho)\cdot\sqrt{1-c_\rho(1-\cos\alpha)}.$$

Using the inequalities $1-\cos\alpha\geqslant\alpha^2/4$ for α close to 0 and $\sqrt{1-y}\leqslant 1-y/2$ for all y, we get

$$\sqrt{1-c_\rho(1-\cos\alpha)} \leqslant \sqrt{1-c_\rho\frac{\alpha^2}{4}} \leqslant 1-c_\rho\frac{\alpha^2}{8}.$$

Returning to f, we have that

$$|f(\rho e^{i\theta})| \leqslant f(\rho)-(f_0+f_1\rho)c_\rho\frac{\alpha^2}{8},$$

i.e.

$$|f(\rho e^{i\theta})| \leqslant f(\rho)-\frac{f_0f_1\rho}{4(f_0+f_1\rho)}\alpha^2.$$

This yields finally

$$\left|\frac{f(\rho e^{i\theta})}{f(\rho)}\right| \leqslant 1-\frac{f_0f_1\rho}{4f(\rho)(f_0+f_1\rho)}\alpha^2.$$

Plugging this inequality into Eq. (7), and using the fact that $1-x\leqslant e^{-x}$, we get

$$|I_2| \leqslant \frac{e^{h(\rho)}}{\sqrt{2\pi h''(\rho)}}\cdot O(\rho\sqrt{h''(\rho)}e^{-C_\rho d\alpha^2}) \quad \text{with } C_\rho = \frac{f_0f_1\rho}{4f(\rho)(f_0+f_1\rho)}. \tag{8}$$

Lemma 3 is used to show that the greatest part of the integral equal to $I_{n,d}$ is negligible, mostly when $n=o(d)$. In this case, $\rho\to 0$ and C_ρ is equivalent to $f_1\rho/(4f_0)$. When n is of order at least d, the saddle point ρ is bounded from below by a strictly positive constant and may go to infinity, and the term C_ρ may become too small for the upper bound obtained by Lemma 3 to be of interest. However, a direct study often gives a useful upper bound, as we shall see in Sections 7.7 and 7.8.

7.5. Proof of Proposition 2

Define ρ_0 as the (exact) solution of $h'(z)=0$, and let $\rho=\rho_0(1+\eta)$, with $\eta=o(1)$. How large can η be, for the approximation of Proposition 1 still to be valid (i.e. not change the main term of the asymptotic expression)? In other words, what is the error when we substitute ρ for ρ_0 in the main term $f(\rho_0)^d/(\rho_0^{n+1}\sqrt{2\pi h''(\rho_0)})$?

We first apply Proposition 1 to obtain an approximate value using the exact saddle point ρ_0:

$$[z^n]\{f(z)^d\}=\frac{f(\rho_0)^d}{\rho_0^{n+1}\sqrt{2\pi d\delta f(\rho_0)}}(1+\text{error terms}).$$

Then we substitute $\rho=\rho_0(1+\eta)$ in the main term and evaluate the error thus introduced: We have that $\rho^{n+1}=\rho_0^{n+1}\mathrm{e}^{(n+1)\eta+\mathrm{O}(n\eta^2)}$. Define $\beta(z)=(f''/f-f'^2/f^2)(z)=\mathrm{d}^2/\mathrm{d}z^2(\log f(z))$ and $\|\beta\|$ as its maximum value for z between ρ and ρ_0: $f(\rho)^d=f(\rho_0)^d\mathrm{e}^{d\eta\Delta f(\rho_0)+\mathrm{O}(d\|\beta\|\eta^2\rho_0^2)}$. Recall that $\Delta f(\rho_0)=(n+1)/d$; we also have $h''(\rho)=h''(\rho_0)(1+\mathrm{o}(1))$. Then

$$\frac{f(\rho)^d}{\rho^{n+1}\sqrt{2\pi h''(\rho)}}=\frac{f(\rho_0)^d}{\rho_0^{n+1}\sqrt{2\pi h''(\rho_0)}}(1+\mathrm{O}(d\|\beta\|\rho_0^2\eta^2)+\mathrm{O}(n\eta^2)+\mathrm{o}(1)). \tag{9}$$

If the substitution of ρ_0 for ρ is to hold, all the error terms we have just introduced must be $\mathrm{o}(1)$:

$$d\|\beta\|\eta^2\rho_0^2=\mathrm{o}(1) \quad \text{and} \quad n\eta^2=\mathrm{o}(1). \tag{10}$$

7.6. *Proof of Theorem 4*

We are now able to prove Theorem 4, which holds when $n,d\to+\infty$ in such a way that $n=\mathrm{o}(d)$. Proposition 2 allows us to use in Proposition 1 an approximate saddle point ρ, defined as the real positive solution of the equation $\Delta f(z)=n/d$, instead of the exact saddle point. By Assumption $\mathscr{A}_1$, $f_1\neq 0$ and the solution ρ of the equation $\Delta f(z)=n/d$ has for approximate value

$$\rho=\frac{f_0 n}{f_1 d}\left(1+\mathrm{O}\left(\frac{n}{d}\right)\right).$$

This solution differs from the exact saddle point ρ_0, defined by $\Delta f(z)=(n+1)/d$, by a factor $\eta=\mathrm{O}(1/n)$. As $n=\mathrm{o}(d)$, we can easily see that $\beta(z)=\mathrm{d}^2/\mathrm{d}z^2(\log f(z))$ is $\mathrm{O}(1)$ for $z=\mathrm{o}(1)$ and that $d\|\beta\|\eta^2\rho_0^2=\mathrm{o}(n\eta^2)$. We finally get that we can use ρ instead of ρ_0 if $n\eta^2=\mathrm{o}(1)$, which holds here. Thus we can substitute $\rho=\rho_0(1+\eta)$ for the exact saddle point ρ_0 in Proposition 1 and get the same approximation of $[z^n]\{f(z)^d\}$.

To derive Theorem 4 from Proposition 1, we now need to know the order of growth of $h''(\rho)$ and of $h''(\rho)$. As

$$z^2h''(z)=z^2d\frac{f''(z)}{f(z)}-d\Delta f(z)^2+n+1,$$

and as, by definition of ρ, $\Delta f(\rho)=n/d$, we have that

$$\rho^2 h''(\rho)=\rho^2 d\frac{f''(\rho)}{f(\rho)}+n\left(1-\frac{n}{d}+\frac{1}{n}\right).$$

Now $\rho^2 df''(\rho)/f(\rho)=\mathrm{O}(\rho^2 d)=\mathrm{O}(n^2/d)$, and

$$\rho^2 h''(\rho)=n\left(1+\frac{1}{n}+\mathrm{O}\left(\frac{n}{d}\right)\right).$$

Hence $\rho^2 h''(\rho)$ is of order exactly n. We next study h''' around ρ: $h'''(z)=d(f'/f)''(z)-2(n+1)/z^3=-2(n+1)/z^3+\mathrm{O}(d)$. This shows that $\rho^3\|h'''\|=\Theta(n)$ and that $\|h'''\|\rho^3$ is again of order exactly n. We use this information to simplify the error terms of Proposition 1:

- We have that $\mathrm{O}(\|h'''\|\rho^3\alpha^3)=\mathrm{O}(n\alpha^3)$.
- The term $\mathrm{O}(\mathrm{e}^{-\rho^2\alpha^2 h''(\rho)/4})$ is $\mathrm{O}(\mathrm{e}^{-an\alpha^2})$ for some constant $a>0$.
- The term $\mathrm{O}(\sqrt{h''(\rho)}\alpha^2\rho\mathrm{e}^{-\rho^2\alpha^2 h''(\rho)/4})$ simplifies into $\mathrm{O}(\alpha^2\sqrt{n}\mathrm{e}^{-an\alpha^2})$.
- The expression $C_\rho=f_0 f_1\rho/(4f(\rho)(f_0+f_1\rho))$ has for approximate value $f_1\rho/(4f_0)(1+\mathrm{o}(1))=n/(4d)(1+\mathrm{o}(1))$ and the term $\mathrm{O}(\rho\sqrt{h''(\rho)}\mathrm{e}^{-C_\rho d\alpha^2})$ is $\mathrm{O}(\sqrt{n}\mathrm{e}^{-an\alpha^2})$ (we can choose the constant a so that it is the same in all the exponential error terms).
- As $\mathrm{e}^{-an\alpha^2}=\mathrm{o}(\sqrt{n}\mathrm{e}^{-an\alpha^2})$ and $\alpha^2\sqrt{n}\mathrm{e}^{-an\alpha^2}=\mathrm{o}(\sqrt{n}\mathrm{e}^{-an\alpha^2})$, the dominant term of the last three error terms in Proposition 1 is $\mathrm{O}(\sqrt{n}\mathrm{e}^{-an\alpha^2})$.

Summing up all these simplifications, we finally get

$$I_{n,d}=\frac{\mathrm{e}^{h(\rho)}}{\sqrt{2\pi h''(\rho)}}(1+\mathrm{O}(n\alpha^3)+\mathrm{O}(\sqrt{n}\mathrm{e}^{-an\alpha^2})). \tag{11}$$

Now we choose α such that $n\alpha^3\to 0$ and $an\alpha^2-(1/2)\log n\to+\infty$. If we take for example $\alpha=\log n/\sqrt{n}$, the error terms in the formula (11) are o(1). We then use the relations $\mathrm{e}^{h(z)}=f^d(z)/z^{n+1}$ and $\rho^2 h''(\rho)=n(1+\mathrm{o}(1))$ to get the final formula:

$$[z^n]\{f^d(z)\}=\frac{f^d(\rho)}{\rho^n\sqrt{2\pi n}}(1+\mathrm{o}(1)).$$

7.7. *Proof of Theorem 5*

We study here the case where $f(z)=\mathrm{e}^{P(z)}$, where $P(z)$ is a polynomial of degree q with positive coefficients; we shall write $P(z)=\sum_{j=0}^{q}p_j z^j$. The coefficient p_1 is null if and only if the coefficient $f_1=[z]\{f\}$ is itself null; thus the condition $\mathscr{A}_1$ on f implies that $p_1\neq 0$. We assume in this part that $n,d\to+\infty$ in such a way that $d=\mathrm{o}(n)$.

The saddle point ρ is defined by the equation $zP'(z)=(n+1)/d$. When $n/d\to+\infty$, it is easy to see that $\rho\to+\infty$. An approximate value is obtained by keeping only the leading term in $zP'(z)$. This gives the equation $qp_q z^q(1+\mathrm{O}(1/z))=(n+1)/d$; hence the saddle point can be written as

$$\rho=\left(\frac{n+1}{qp_q d}\right)^{1/q}\left(1+\mathrm{O}\left(\frac{d^{1/q}}{n^{1/q}}\right)\right).$$

Our next step is an evaluation of the order of growth of the second and third derivatives of $h(z)=dP(z)-(n+1)\log z$ at point ρ. We have that $z^2h''(z)=dz^2P''(z)+n+1$, hence

$$\begin{aligned}\rho^2 h''(\rho)&=dq(q-1)p_q\rho^q+\mathrm{O}(\mathrm{d}\rho^{q-1})+n+1\\&=q(n+1)+\mathrm{O}(d\rho^{q-1})\\&=q(n+1)\left(1+\mathrm{O}\left(\frac{d^{1/q}}{n^{1/q}}\right)\right).\end{aligned}$$

Hence $\rho^2h''(\rho)$ is of order exactly n. A similar computation shows that $\rho^3h'''(\rho)$ is also of order n.

Proposition 1, as given above, holds for a very limited range of n and d, but we can adapt its proof and adjust the error term from Lemma 3 containing C_ρ. The outline of the proof is the same: We split $\mathbf{I}_{n,d}=[z^n]\{f^d(z)\}$ in two integrals I_1 and I_2, with I_1 representing the integral close to the saddle point and I_2 the rest of the integral. Then Lemma 2 gives an evaluation of I_1 and we just have to modify the part dealing with I_2 (the value C_ρ given by Lemma 3 is here exponentially small and would give no useful bound). This requires the computation of an upper bound on the modulus of $(1/2\mathrm{i}\pi)\int\exp(h(z))\,\mathrm{d}z$, where the integral path is for $z=\rho\mathrm{e}^{\mathrm{i}\theta}$ with $\alpha\leqslant|\theta|\leqslant\pi$:

$$I_2=\frac{1}{2\mathrm{i}\pi}\int_{z=\rho\mathrm{e}^{\mathrm{i}\theta},\,\alpha\leqslant|\theta|\leqslant\pi} f(z)^d\frac{\mathrm{d}z}{z^{n+1}}$$

and

$$|I_2|\leqslant\frac{f(\rho)^d}{2\pi\rho^n}\int_{\alpha\leqslant|\theta|\leqslant\pi}\left|\frac{f(\rho\mathrm{e}^{\mathrm{i}\theta})}{f(\rho)}\right|^d d\theta.$$

Now

$$\frac{|f(\rho\mathrm{e}^{\mathrm{i}\theta})|}{f(\rho)}=|\mathrm{e}^{P(\rho\mathrm{e}^{\mathrm{i}\theta})-P(\rho)}|=\mathrm{e}^{\Re\{P(\rho\mathrm{e}^{\mathrm{i}\theta})-P(\rho)\}}.$$

But $\Re\,\{P(\rho\mathrm{e}^{\mathrm{i}\theta})-P(\rho)\}=\sum_{1\leqslant j\leqslant q}p_j\rho^j(\cos j\theta-1)\leqslant -p_1\rho(1-\cos\theta)$. As we assume in Theorem 5 that the coefficient p_1 is non-null, we have a bound on the modulus of $f(\rho\mathrm{e}^{\mathrm{i}\theta})/f(\rho)$, for a suitable strictly positive constant a:

$$\frac{|f(\rho\mathrm{e}^{\mathrm{i}\theta})|}{f(\rho)}\leqslant\mathrm{e}^{-a\rho\alpha^2}\quad\text{for }\alpha\leqslant|\theta|\leqslant\pi.$$

Hence

$$|I_2|\leqslant\frac{\mathrm{e}^{h(\rho)}}{\sqrt{2\pi h''(\rho)}}\mathrm{e}^{-ad\rho\alpha^2}\mathrm{O}(\sqrt{n}).$$

The saddle point approximation coming from this bound and from Lemma 2 then gives error terms

$$O(\alpha^2\sqrt{\rho^2 h''}\,e^{-\rho^2\alpha^2 h''/4})+O(\|h'''\|\rho^3\alpha^3)+O(e^{-\rho^2\alpha^2 h''/4})+O(\sqrt{n}\,e^{-ad\rho\alpha^2}).$$

Adjusting the constant a if necessary, we can simplify these terms into

$$O(\alpha^2\sqrt{n}\,e^{-an\alpha^2})+O(n\alpha^3)+O(e^{-an\alpha^2})+O(\sqrt{n}\,e^{-ad\alpha^2(n/d)^{1/q}}).$$

Now if the last term is o(1), then $d\alpha^2(n/d)^{1/q}\to+\infty$; for $d=o(n)$ this implies that $n\alpha^2\to+\infty$ and that the first and the third error terms are o(1). Thus the conditions required for the error terms to be o(1) are that $n\alpha^3\to 0$ and that $d\alpha^2(n/d)^{1/q}$ be of order at least $\log n$ (we can then multiply α by a constant in such a way that $ad\alpha^2(n/d)^{1/q}\geqslant\log n$, thus ensuring that the last term of the error is smaller than $\sqrt{n}\,e^{-\log n}=1/\sqrt{n}$). For both these conditions to hold, n must not grow too fast with respect to d. We find that n and d must satisfy

$$(\log n)^{3q}n^{2q-3}=o(d^{3(q-1)}).$$

This condition holds, for example, as soon as $n=O(d^{(3q-3-\varepsilon)/(2q-3)})$, for any $\varepsilon>0$. More precisely, a value for α such that the error terms are o(1) is $\alpha=(\log n)^{1/4}n^{-1/6-1/4q}d^{(1-q)/4q}$.

We now check that we can substitute an approximate saddle point for the exact value $\rho_0=(\Delta f)^{-1}((n+1)/d)$ we have been using. To use Proposition 2, we have to know the order of growth of $\beta=(f''/f)-(f'/f)^2$, evaluated around the point ρ_0. For $f=e^P$, we have that $\beta(z)=P''(z)$, and $\beta(\rho_0)$ is of order exactly ρ_0^{q-2}. Now $P''(z)$ is a polynomial with positive coefficients and its maximum near ρ_0 is of the same order as $P''(\rho_0)$. Hence the condition $d\|\beta\|\rho_0^2\eta^2=o(1)$ simplifies into $n\eta^2=o(1)$. The other condition derived from Proposition 2 also reduces to $n\eta^2=o(1)$. To sum up, the saddle point approximation still holds whenever we choose a saddle point $\rho=\rho_0(1+\eta)$ such that $\eta=o(1/\sqrt{n})$. Choosing the solution of $\Delta f(z)=n/d$ moves it away from ρ_0 by $\eta=O(1/n)=o(1/\sqrt{n})$, so it obviously works.

7.8. Proof of Theorem 6

This concerns the case where f is meromorphic, with a unique singularity on its circle of convergence, and where $d=o(n)$ with $n,d\to+\infty$. We can normalize f so that its pole closest to the origin is 1; let p be its order. Then we can write $f(z)=g(z)/(1-z)^p$, with g a function analytic for $|z|\leqslant 1$. The saddle point is defined by the equation

$$\Delta g(z)+\frac{pz}{1-z}=\frac{n+1}{d}.$$

When $z\to 1$ in such a way that $|z|\leqslant 1$, $\Delta g(z)$ is bounded and $pz/(1-z)\to+\infty$. Hence, for $n/d\to+\infty$, the saddle point is close to 1. More precisely, we have that

$$\rho=1-\frac{pd}{n+1}+O\left(\frac{d^2}{n^2}\right).$$

Now $h(z)=d\log f(z)-(n+1)\log z=d\log g(z)-pd\log(1-z)-(n+1)\log z$ and

$$h''(z)=d\left(\frac{g'}{g}\right)'(z)+\frac{pd}{(1-z)^2}+\frac{n+1}{z^2}.$$

For z close to 1, the main contribution comes from the second term of h''. Then $h''(\rho)=n^2/(pd)(1+o(1))$; hence $h''(\rho)$ is strictly positive. We can similarly prove that $h'''(\rho)=\Theta(n^3/d^2)$. We then follow the proof of Proposition 1; Lemma 2 applies without modification. However, we do not use Lemma 3 (which is proved without assumptions on the function f, but is mainly intended for a saddle point of sufficiently small modulus) but study directly the integral. As the function g has positive coefficients, we have that

$$\frac{|f(\rho e^{i\theta})|}{f(\rho)}=\frac{|g(\rho e^{i\theta})|}{g(\rho)}\left(\frac{1-\rho}{|1-\rho e^{i\theta}|}\right)^p\leqslant\left(\frac{1-\rho}{|1-\rho e^{i\theta}|}\right)^p.$$

Now $|1-\rho e^{i\theta}|=(1-2\rho\cos\theta+\rho^2)^{1/2}$ and a lower bound on this will give an upper bound on $|f(\rho e^{i\theta})/f(\rho)|$.

If we assume that α is such that $\alpha/(1-\rho)\to 0$, i.e. that $\alpha n/d\to 0$, we have that, for $\alpha\leqslant|\theta|\leqslant\pi$ and for some strictly positive constant a,

$$(1-2\rho\cos\theta+\rho^2)^{1/2}\geqslant(1-2\rho\cos\alpha+\rho^2)^{1/2}\geqslant(1-\rho)(1+a\rho\alpha^2/(1-\rho)^2).$$

Hence, adjusting a if necessary,

$$\frac{|1-\rho|}{|1-\rho e^{i\theta}|}\leqslant 1-a\frac{\alpha^2}{(1-\rho)^2},$$

and

$$\frac{|f(\rho e^{i\theta})|}{f(\rho)}\leqslant\left(1-a\frac{\alpha^2}{(1-\rho)^2}\right)^p.$$

This shows that the error term coming from the integral $(1/2i\pi)\int e^{h(z)}dz$, for $z=\rho e^{i\theta}$ and $\alpha\leqslant|\theta|\leqslant\pi$, is $e^{h(\rho)}O(\rho e^{-apd\alpha^2/(1-\rho)^2})$. Factoring out the leading term $e^{h(\rho)}/\sqrt{2\pi h''(\rho)}$, we see that the integral far from the saddle point contributes an error term $O(\rho\sqrt{h''}e^{-apd\alpha^2/(1-\rho)^2})$, i.e. $O(n/\sqrt{d}e^{-b\alpha 2n^2/d})$ (recall that $\rho=1-pd/n(1+o(1))$). From this and from the error terms of Lemma 2, we find that the saddle point approximation holds if we can find some $\alpha>0$ such that

$$\alpha n/d\to 0,$$

$$\frac{n}{\sqrt{d}}e^{-b\alpha^2n^2/d}\to 0,$$

$$\alpha^3n^3/d^2\to 0.$$

The third condition implies the first one when $d\to+\infty$, so we are left with the two requirements that $\alpha^3n^3/d^2\to 0$ and that α^2n^2/d is of order at least $\log(n/\sqrt{d})$ (here again, we multiply α by a constant factor if desired). For such an α to exist, n and

d must be such that $\log(n/\sqrt{d}) = o(d^{1/3})$. If this holds, we then choose $\alpha = (\log(n/\sqrt{d}))^{1/4} d^{7/12}/n$, and the error terms become $o(1)$.

When we want to use an approximate saddle point, from Proposition 2 we need an estimation of $\beta(z) = d^2/dz^2(\log g(z)) + p/(1-z)^2$ in the vicinity of the saddle point, which is close to 1. But for z near 1, $\beta(z) = p/(1-z)^2(1+o(1))$; hence $\beta(\rho) = O(n^2/d^2)$. Here again the error terms introduced by integrating on a circle of radius $\rho(1+\eta)$ are $O(n\eta^2)$ and $O(n^2\eta^2/d)$, and they become $o(1)$ for $\eta = o(\sqrt{d}/n)$. Thus we can substitute the solution of the equation $\Delta f(z) = n/d$ to the exact saddle point, without changing the main term of the approximation.

7.9. *Proof of Proposition 3*

In this section we introduce a function ψ that satisfies Assumption $\mathcal{A}_2$ of Section 4. To compute $I_{n,d} = [z^n]\{f^d\psi\}$, we define

$$h(z) = d\log f(z) + \log\psi(z) - (n+1)\log z.$$

The outline of the proof is the same as that of Proposition 1 in Section 7.2, and we only indicate where it differs. The saddle point ρ is defined by $h'(z) = 0$, i.e. by the equation

$$\Delta f(z) + \Delta\psi(z)/d = (n+1)/d. \tag{12}$$

For $\alpha \in]0, \pi[$, we divide the integral in two parts: $I_{n,d} = I_1 + I_2$; as in Section 7.2, I_1 is the integral on $|\theta| \leqslant \alpha$ and I_2 is the integral for $\alpha \leqslant |\theta| \leqslant \pi$. The value of h'' at the point ρ defined by Eq. (12) is

$$h''(\rho) = d\delta f(\rho) + \delta\psi(\rho).$$

The values of $\delta f(\rho)$ and of $\delta\psi(\rho)$ are strictly positive when the initial functions f and ψ have real positive coefficients; this shows that $h''(\rho)$ also is strictly positive. The integral I_1 can then be approximated using Lemma 2 and we obtain again the approximation

$$I_1 = \frac{e^{h(\rho)}}{\sqrt{2\pi h''\rho)}}(1 + O(\|h'''\|\rho^3\alpha^3) + O(\sqrt{h''(\rho)}\alpha^2\rho e^{-(\rho^2/4)\alpha^2 h''(\rho)}) + O(e^{-(\rho^2/4)\alpha^2 h''(\rho)})).$$

As for I_2, we have an upper bound

$$|I_2| \leqslant \frac{e^{h(\rho)}}{2\pi}\rho\int_{\alpha\leqslant|\theta|\leqslant\pi}\left|\frac{f(\rho e^{i\theta})}{f(\rho)}\right|^d \cdot \left|\frac{\psi(\rho e^{i\theta})}{\psi(\rho)}\right| d\theta.$$

The function ψ has positive coefficients and satisfies $|\psi(\rho e^{i\theta})| \leqslant |\psi(\rho)|$, hence

$$|I_2| \leqslant \frac{e^{h(\rho)}}{2\pi}\rho\int_{\alpha\leqslant|\theta|\leqslant\pi}\left|\frac{f(\rho e^{i\theta})}{f(\rho)}\right|^d d\theta.$$

Now Lemma 3 can be applied to the integral on the right-hand side; this gives an upper bound on I_2. Finally, the approximation of I_1 and the bound on I_2 give the desired evaluation of $I_{n,d}$.

7.10. Proof of Proposition 4

Following the notations of Section 7.5, we define ρ_0 as the solution of $h'(z)=0$ and let $\rho=\rho_0(1+\eta)$, with $\eta=o(1)$. The saddle point ρ_0 in Proposition 3 is defined by the equation $\Delta f(z)+\Delta\psi(z)/d=(n+1)/d$. What conditions must be satisfied by ψ and η if we want to substitute ρ for ρ_0 in Proposition 3 and still get a valid approximation?

As was the case for f, we have that $\psi(\rho)=\prod_j g_j(\rho)^{d_j}$ and that, for each j,

$$g_j(\rho)^{d_j}=g_j(\rho_0)^{d_j}\exp(d_j\eta\Delta g_j(\rho_0)+\mathrm{O}(d_j\beta_j(\rho_0)\eta^2\rho_0^2)),$$

with $\beta_j(z)=\mathrm{d}^2(\log g_j)/\mathrm{d}z^2$. Hence

$$\psi(\rho)=\psi(\rho_0)\mathrm{e}^{\eta\Delta\psi(\rho_0)+\sum_j\mathrm{O}(d_j\beta_j(\rho_0)\eta^2\rho_0^2)}.$$

We recall (see Section 7.5) that $\rho^{n+1}=\rho_0^{n+1}\mathrm{e}^{(n+1)\eta+\mathrm{O}(n\eta^2)}$ and that $f(\rho)^d=f(\rho_0)^d\mathrm{e}^{d\eta\Delta f(\rho_0)+\mathrm{O}(d\|\beta\|\eta^2\rho_0^2)}$. By definition of ρ_0, $d\Delta f(\rho_0)+\Delta\psi(\rho_0)-(n+1)=0$ and we have that

$$\frac{f(\rho)^d\psi(\rho)}{\rho^{n+1}\sqrt{2\pi h''(\rho)}}=\frac{f(\rho_0)^d\psi(\rho_0)}{\rho_0^{n+1}\sqrt{2\pi h''(\rho_0)}}$$

$$\times\left(1+\mathrm{O}(d\|\beta\|\rho_0^2\eta^2)+\mathrm{O}(n\eta^2)+\sum_j\mathrm{O}(d_j\|\beta_j\|\eta^2\rho_0^2)+o(1)\right).$$

7.11. Proof of Theorem 8

We recall that Theorem 8 gives an approximation of $[z^n]\{f^d(z)\psi(z)\}$ when $n=\Theta(d)$. We do not give a detailed proof of Theorem 8, but instead indicate the main points where it differs from the proof of Theorem 4, which is very similar. The main difference is that the saddle point belongs to a fixed and finite interval $[\rho_1,\rho_2]$ with $\rho_1>0$. We define

$$h(z)=d\log f(z)+\sum_{j=1}^{p}d_j\log g_j(z)-(n+1)\log z.$$

The real positive solution ρ of the equation $h'(z)$ is again unique. As $h''(\rho)=\mathrm{d}\delta f(\rho)+\sum_j\mathrm{O}(d_j)$ and as the function $\delta f(z)$ takes strictly positive values for real positive arguments, we can check that $h''(\rho)=\Theta(d)$. Similarly, h'' is of order d around ρ. Hence Lemma 2 holds with error terms

$$\mathrm{O}(\alpha^2\sqrt{d}\mathrm{e}^{-Cd\alpha^2})+\mathrm{O}(d\alpha^3)+\mathrm{O}(\mathrm{e}^{-Cd\alpha^2}).$$

Simplifying, we see that the conditions for these error terms to be o(1) are that $d\alpha^2 \to +\infty$ and that $d\alpha^3 \to 0$. Lemma 3 can be easily modified with the following remark: by Assumption $\mathscr{A}_2$, the function $\psi(z)=\prod_j g_j^{d_j}(z)$ has positive coefficients, hence $|\psi(\rho e^{i\theta})|/\psi(\rho) \leqslant 1$. Then the result of Lemma 3 applies with a factor $C_\rho=\Theta(1)$; hence we get from its error term the condition that $\sqrt{d}e^{-C_\rho d\alpha^2} \to 0$. Putting all this together, we see that choosing $\alpha=\log n/\sqrt{n}$ gives the theorem.

7.12. Proof of Theorem 9

In this part, we deal with the case $n, d \to +\infty$ and $n=o(d)$. The proof outline should now be familiar to the reader and we shall limit ourself to the main points. We shall first prove that the saddle point approximation holds when choosing for radius of the integration circle the exact saddle point ρ_0, which is the solution of the equation $\Delta f(z)+\Delta\psi/d=(n+1)/d$. Then we shall use Proposition 4 to show that we can use the solution of the approximate equation $\Delta f(z)=n/d$ instead of ρ_0.

The exact saddle point ρ_0 is defined by the equation

$$\Delta f(z)+\frac{1}{d}\sum_{j=1}^{p} d_j \Delta g_j(z)=\frac{n+1}{d}.$$

When $z \to 0$, all the terms $\Delta f(z)$ and $\Delta g_j(z)$ are $O(z)$, and ρ_0, the real positive solution of this equation, is close to the solution of the equation $\Delta f(z)=(n+1)/d$ as soon as each d_j is $o(d)$. Hence ρ_0 is equal to $f_0(n+1)/(f_1 d)(1+o(1))$, and we have that $\rho_0^2 h''(\rho_0^2)=n(1+o(1))$. Proposition 3 holds; its last three error terms simplify into $O(\sqrt{n}e^{-n\alpha^2})$. As was the case in the proof of Theorem 4, $\|h'''\|$ is of order $\Theta(d^3/n^2)$ around the saddle point, and $\rho_0^3\|h'''\|=\Theta(n)$; hence $O(\|h'''\|\rho^3\alpha^3)=O(n\alpha^3)$. As before, we choose $\alpha=\log n/\sqrt{n}$; then $n\alpha^3 \to 0$ and $n\alpha^2/\log n \to +\infty$; the error terms in the approximate expression of $I_{n,d}$ are o(1) and we get that

$$[z^n]\{f(z)^d\psi(z)\}=\frac{f(\rho_0)^d\psi(\rho_0)}{\rho_0^n\sqrt{2\pi n}}(1+o(1)).$$

We now study when this approximation holds for a simpler saddle point $\rho=\rho_0(1+\eta)$, defined by the approximate equation $\Delta f(z)=n/d$. From Proposition 4, and taking into account the fact that ρ_0 is of order n/d, we have to verify that $n\eta^2=o(1)$, that $\beta(\rho_0)n^2\eta^2/d=o(1)$ and that, for each j, $d_j\beta_j(\rho_0)n^2\eta^2/d^2=o(1)$. When $d_j \leqslant d$ for all j, $\beta(\rho_0)$ and all the $\beta_j(\rho_0)$ have order O(1) and the former conditions collapse into $n\eta^2=o(1)$; hence we have to check that $\eta=o(1/\sqrt{n})$. But we have that $\Delta f(\rho)=\Delta f(\rho_0)+\rho_0\eta(\Delta f)'(z)$ for some z close to ρ_0, and that, for each j, $\Delta g_j(\rho)=O(\rho_0)$. Hence $\Delta f(\rho)+\Delta\psi(\rho)=\Delta f(\rho_0)+\rho_0\eta(\Delta f)'(z)+\sum_{1\leqslant j\leqslant p}O(\rho_0 d_j/d)$. This shows that $\eta=O(1/n)+\sum_j O(d_j/d)$. Thus the substitution of ρ for ρ_0 is valid if and only if, for each j, $d_j=o(d/\sqrt{n})$.

7.13. Proofs of Corollaries 2 and 3

Case where $n=o(\sqrt{d})$: We shall further simplify Theorems 4 and 9 when $n=o(\sqrt{d})$. We recall that, for $n=o(d)$, $\Delta f(z)=(f_1/f_0)z+O(z^2)$, and that the (approximate) saddle point is $\rho=f_0 n/(f_1 d)(1+O(n/d))$. Hence $\rho^n=(f_0 n/(f_1 d))^d(1+O(n^2/d)+O(1/n))$, $f(\rho)=f_0(1+n/d+O(n^2/d^2))$ and $f(\rho)^d=f_0^d e^n(1+O(n^2/d))$. When $n=o(\sqrt{d})$, the error terms are o(1) and we get

$$[z^n]\{f^d(z)\}=\frac{f_0^{d-n}}{\sqrt{2\pi n}}\cdot\left(\frac{ef_1 d}{n}\right)^n(1+o(1)).$$

Extension when $n\neq o(\sqrt{d})$ but $n=o(d^{2/3})$: We first improve on the approximation of the saddle point: $\Delta f(z)=f_1 z/f_0\cdot(1+(2f_2/f_1-f_1/f_0)z+O(z^2))$. The solution ρ of $\Delta f(z)=n/d$ is then

$$\rho=\frac{f_0 n}{f_1 d}\left(1+\left(1-2\frac{f_0 f_2}{f_1^2}\right)\frac{n}{d}+O\left(\frac{n^2}{d^2}\right)\right),$$

and we have

$$\rho^n=\left(\frac{f_0 n}{f_1 d}\right)^n\exp\left(\left(1-2\frac{f_0 f_2}{f_1^2}\right)\frac{n^2}{d}\right)\left(1+O\left(\frac{n^3}{d^2}\right)\right).$$

The approximate value of ρ also gives

$$f(\rho)=f_0\left(1+\frac{n}{d}+\left(1-\frac{f_0 f_2}{f_1^2}\right)\frac{n^2}{d^2}+O\left(\frac{n^3}{d^3}\right)\right),$$

hence

$$f(\rho)^d=f_0^d e^n\exp\left(\left(\frac{1}{2}-\frac{f_0 f_2}{f_1^2}\right)\frac{n^2}{d}\right)\left(1+O\left(\frac{n^3}{d^2}\right)\right).$$

Putting all this together, and assuming that $n=o(d^{2/3})$, we get

$$[z^n]\{f^d(z)\}=\frac{f_0^{d-n}}{\sqrt{2\pi n}}\left(\frac{ef_1 d}{n}\right)^n\exp\left(\frac{n^2}{d}\left(\frac{f_0 f_2}{f_1^2}-\frac{1}{2}\right)\right)(1+o(1)).$$

If there is a factor ψ: In this case, we merely have to see that $\psi(\rho)=\psi(0)(1+o(1))$. This comes from

$$\psi(\rho)=\psi(0)\left(1+O\left(\rho\sum_{j=1}^{p}d_j\right)\right)=\left(1+O\left(n\sum_{j=1}^{p}d_j/d\right)\right).$$

As long as each d_j is $o(d/n)$, the error term is o(1) and

$$\psi(\rho)=\psi(0)\left(1+O\left(\rho\left\|\frac{\psi'}{\psi}\right\|\right)\right)=\psi(0)(1+o(1)).$$

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